

# THE EFFECT OF PERTURBATIONS OF FRAMES AND FUSION FRAMES ON THEIR REDUNDANCIES

A. RAHIMI, G. ZANDI AND B. DARABY

**ABSTRACT.** An interesting question about the perturbed sequences is: when do they inherit the properties of the original one? An elegant relation between frames (fusion frames) and their perturbations is the relation of their redundancies. In this paper, we investigate these relationships. Also, we express the redundancy of frames (fusion frames) in terms of the cosine angle between some subspaces.

## 1. INTRODUCTION

Introduced in 1952 by Duffin and Schaeffer [10], frames as an extension of the concept of orthonormal bases allowing for redundancy, while still maintaining stability properties, are today an standard notion in mathematics and engineering. Applications range from more theoretical problems such as the Kadison- Sineger problem [7, 4] and tensor decomposition [20] over questions inspired by applications such as sparse approximation theory [17], quantum mechanics [11], inverse scattering problems [18] and wave packet systems on finite cyclic groups, fields and Abelian groups [12, 13, 14]. Recently, due to both necessities from applications and theoretical goals, generalizations of this framework have been developed, first fusion frames [6], then operator-valued frames [16] and g-frames [23]. While frames and their extensions are by definition stable in the sense of their analysis operator being continuous and bounded below, applications require stronger forms of stability. Of particular importance is robustness with respect to perturbations which is typically regarded as a property of the associated analysis operators. More precisely, in the frame setting, say one would refer to a frame  $\Psi = \{\psi_i\}_{i \in I}$  as being a  $\mu$ -perturbation of  $\Phi = \{\varphi_i\}_{i \in I}$ , if  $\|T_\Phi - T_\Psi\| \leq \mu$ , where  $T$  is the analysis operator. In the situation of frames, this is a well-studied subject (see, e.g., [1, 9]).

An interesting question about the perturbed sequences is: when do they inherit the properties of the original one. For instance, it is known that if  $\{\varphi_i\}_{i \in I}$  is a Riesz sequence with lower bound  $A$ , the perturbed sequence  $\{\psi_i\}_{i \in I}$  is also a Riesz sequence when  $\mu < \sqrt{A}$ , [8]. An elegant relation between a frame and its perturbations is the relation of their redundancies. As

---

2010 *Mathematics Subject Classification.* Primary 42C40; Secondary 41A58, 47A58.

*Key words and phrases.* Fusion Frame, Redundancy Function, Perturbation, Infimum Cosine Angle.

we know, redundancy appears as a mathematical concept and as a methodology for signal processing. Recently, the ability of redundant systems to provide sparse representations has been extensively exploited [3]. In fact, frame theory is entirely based on the notion of redundancy. This idea can be interesting to find the relation between redundancy of a frames (fusion frames) and the redundancy of its perturbations. The notion of the redundancy of a frame were introduced in [2] and explained by authors in [22] for fusion frames. For general frames (also fusion frames), redundancy should give us information, for instance, about orthogonality and tightness of the frame, about the maximal number of spanning sets and the minimal number of linearly independent sets our frame can be divided into, and about robustness with respect to erasures. Ideally, in the case of a unit-norm tight frame, upper and lower redundancies coincide and equal the customary measure of redundancy  $\frac{N}{n}$ , where  $N$  denotes the number of frame vectors and  $n$  the dimension of underlying Hilbert space.

## 2. PRELIMINARIES

In this chapter, we state some of the basic definitions and theorems that are needed in chapter 3.

**2.1. A Review of Frames.** Given a separable Hilbert space  $\mathcal{H}$  with norm  $\|\cdot\|$ , a sequence  $\Phi = \{\varphi_i\}_{i \in I}$ , where  $I$  is a countable index set, is a frame for  $\mathcal{H}$  if there exist  $0 < A \leq B$  such that

$$A\|x\|^2 \leq \sum_{i \in I} |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2$$

for all  $x \in \mathcal{H}$ . The constants  $A, B$  are called frame bounds. If only the right inequality is satisfied, we say that  $\Phi = \{\varphi_i\}_{i \in I}$  is a *Bessel sequence* with Bessel bound  $B$ . To every Bessel sequence  $\Phi = \{\varphi_i\}_{i \in I}$ , we associate the analysis operator  $T_\Phi : \mathcal{H} \rightarrow l^2(I)$  defined by  $T_\Phi x = \{\langle x, \varphi_i \rangle\}_{i \in I}$  for  $x \in \mathcal{H}$ , and the synthesis operator  $T_\Phi^* : l^2(I) \rightarrow \mathcal{H}$  given by  $T_\Phi^* c = \sum_{i \in I} c_i \varphi_i$  for  $c = \{c_i\}_{i \in I} \in l^2(I)$ .

If  $\Phi = \{\varphi_i\}_{i \in I}$  is a frame for  $\mathcal{H}$ , the frame operator defined by

$$S_\Phi : \mathcal{H} \rightarrow \mathcal{H}, \quad S_\Phi x = \sum_{i \in I} \langle x, \varphi_i \rangle \varphi_i, \quad x \in \mathcal{H}.$$

This operator is bounded, positive and invertible. A sequence  $\{\varphi_i\}_{i \in I} \subseteq \mathcal{H}$  is a *Riesz basis* for  $\mathcal{H}$  if it is complete in  $\mathcal{H}$  and if there exist  $0 < A \leq B$  such that for every finite scalar sequence  $c = \{c_i\}_{i \in I}$  one has

$$A\|c\|_{l^2}^2 \leq \left\| \sum_{i \in I} c_i \varphi_i \right\|^2 \leq B\|c\|_{l^2}^2,$$

the constants  $A$  and  $B$  are called *Riesz bounds*.

In this paper, we will work with perturbed sequences. More precisely we will use the following notion of perturbation [15].

**Definition 2.1.** Let  $\Phi = \{\varphi_i\}_{i \in I}$  be a sequence in  $\mathcal{H}$  and  $\mu > 0$ . We say that a sequence  $\Psi = \{\psi_i\}_{i \in I}$  in  $\mathcal{H}$  is a  $\mu$ -perturbation of  $\Phi = \{\varphi_i\}_{i \in I}$  if for every finite scalar sequence  $c = \{c_i\}$ ,

$$\left\| \sum c_i(\varphi_i - \psi_i) \right\| \leq \mu \|c\|_{l^2}.$$

The idea of finding the relation between redundancies of a frame (fusion frame) and its perturbations should be interesting. At the first, we review the definition of the redundancy function for finite frames.

**Definition 2.2.** [2] Let  $\Phi = \{\varphi_i\}_{i=1}^N$  be a frame for a finite dimensional Hilbert space  $\mathcal{H}^n$ . For each  $x \in \mathbb{S} = \{x \in \mathcal{H} : \|x\| = 1\}$ , the *redundancy function*

$$\mathcal{R}_\Phi : \mathbb{S} \rightarrow \mathbb{R}^+$$

is defined by

$$\mathcal{R}_\Phi(x) = \sum_{i=1}^N \|P_{\langle \varphi_i \rangle}(x)\|^2, \quad x \in \mathbb{S}$$

where  $\langle \varphi_i \rangle$  denotes the span of  $\varphi_i \in \mathcal{H}$  and  $P_{\langle \varphi_i \rangle}$  denotes the orthogonal projection onto  $\langle \varphi_i \rangle$ . The *upper redundancy* of  $\Phi$  is defined by

$$\mathcal{R}_\Phi^+ = \sup_{x \in \mathbb{S}} \mathcal{R}_\Phi(x),$$

and the *lower redundancy* of  $\Phi$  by

$$\mathcal{R}_\Phi^- = \inf_{x \in \mathbb{S}} \mathcal{R}_\Phi(x).$$

The frame  $\Phi$  has *uniform redundancy* if  $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+$ .

For more studies and details on the redundancy of frames, we refer to [2]. The following theorem states conditions for perturbed sequences to be a frame or a Riesz basis [8].

**Theorem 2.3.** [8] Let  $\Phi = \{\varphi_i\}_{i=1}^N$  be a sequence in  $\mathcal{H}$  and assume that  $\Psi = \{\psi_i\}_{i=1}^N$  is a  $\mu$ -perturbation of  $\Phi$ . Then the following holds:  
If  $\Phi$  is a frame (Riesz basis) for  $\mathcal{H}$  with frame (Riesz) bounds  $0 < A \leq B$  and  $\mu < \sqrt{A}$ , then  $\Psi$  is a frame (Riesz basis) for  $\mathcal{H}$  with frame (Riesz) bounds  $A(1 - \frac{\mu}{\sqrt{A}})^2$ ,  $B(1 + \frac{\mu}{\sqrt{B}})^2$ .

**2.2. A Review of Fusion Frames.** In this subsection, we briefly recalling the basic definitions and notations of *fusion frames*. We wish to mention that the definition of a fusion frame and its associated fusion frame operator already appeared in [6] under the label “frame of subspaces”. Besides, we set the definition of the redundancy function for fusion frames which introduced by authors in [22].

**Definition 2.4.** Let  $\mathcal{H}$  be a Hilbert space and  $I$  be a (finite or infinite) countable index set. Assume that  $\{W_i\}_{i \in I}$  be a sequence of closed subspaces in  $\mathcal{H}$  and  $\{v_i\}_{i \in I}$  be a family of weights, i.e.,  $v_i > 0$  for all  $i \in I$ . We say that

the family  $\mathcal{W} = \{(W_i, v_i)\}_{i \in I}$  is a *fusion frame* or a *frame of subspaces* with respect to  $\{v_i\}_{i \in I}$  for  $\mathcal{H}$  if there exist constants  $0 < A \leq B < \infty$  such that

$$A\|x\|^2 \leq \sum_{i \in I} v_i^2 \|P_{W_i}(x)\|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H},$$

where  $P_{W_i}$  denotes the orthogonal projection onto  $W_i$ , for each  $i \in I$ . The fusion frame  $\mathcal{W} = \{(W_i, v_i)\}_{i \in I}$  is called *tight* if  $A = B$  and *Parseval* if  $A = B = 1$ . If all  $v_i$ 's take the same value  $v$ , then  $\mathcal{W}$  is called *v-uniform*. Moreover,  $\mathcal{W}$  is called an *orthonormal fusion basis* for  $\mathcal{H}$  if  $\mathcal{H} = \bigoplus_{i \in I} W_i$ . If  $\mathcal{W} = \{(W_i, v_i)\}_{i \in I}$  possesses an upper fusion frame bound but not necessarily a lower bound, we call it a *Bessel fusion sequence* with Bessel fusion bound  $B$ . The normalized version of  $\mathcal{W}$  is obtained when we choose  $v_i = 1$  for all  $i \in I$ . Note that we use this term merely when  $\{(W_i, 1)\}_{i \in I}$  formes a fusion frame for  $\mathcal{H}$ .

Without loss of generality, we may assume that the family of weights  $\{v_i\}_{i \in I}$  belongs to  $\ell_+^\infty(I)$ . The analysis, synthesis and fusion frame operator are defined as follows.

**Notation:** For any family  $\{\mathcal{H}_i\}_{i \in I}$  of Hilbert spaces, we use the representation space

$$\left( \sum_{i \in I} \oplus \mathcal{H}_i \right)_{\ell_2} = \left\{ \{f_i\}_{i \in I} : f_i \in \mathcal{H}_i, \sum_{i \in I} \|f_i\|^2 < \infty \right\}$$

with inner product

$$\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle, \quad \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \in \left( \sum_{i \in I} \oplus \mathcal{H}_i \right)_{\ell_2}$$

and

$$\|\{f_i\}_{i \in I}\| := \sqrt{\sum_{i \in I} \|f_i\|^2}.$$

It is easy to show that  $(\sum_{i \in I} \oplus \mathcal{H}_i)_{\ell_2}$  is a Hilbert space.

**Definition 2.5.** Let  $\mathcal{W} = \{(W_i, v_i)\}_{i \in I}$  be a fusion frame for  $\mathcal{H}$ . The *synthesis operator*  $T_{\mathcal{W}} : (\sum_{i \in I} \oplus W_i)_{\ell_2} \rightarrow \mathcal{H}$  is defined by

$$T_{\mathcal{W}}(\{f_i\}_{i \in I}) = \sum_{i \in I} v_i f_i, \quad \{f_i\}_{i \in I} \in \left( \sum_{i \in I} \oplus W_i \right)_{\ell_2}.$$

In order to map a signal to the representation space, i.e., to Analyze it, the *analysis operator*  $T_{\mathcal{W}}^*$  is employed which is defined by

$$T_{\mathcal{W}}^* : \mathcal{H} \rightarrow \left( \sum_{i \in I} \oplus W_i \right)_{\ell_2} \quad \text{with} \quad T_{\mathcal{W}}^*(f) = \{v_i P_{W_i}(f)\}_{i \in I},$$

for any  $f \in \mathcal{H}$ . The *fusion frame operator*  $S_{\mathcal{W}}$  for  $\mathcal{W}$  is defined by

$$S_{\mathcal{W}}(f) = T_{\mathcal{W}} T_{\mathcal{W}}^*(f) = \sum_{i \in I} v_i^2 P_{W_i}(f), \quad f \in \mathcal{H}.$$

It follows from [6] that for each fusion frame, the operator  $S_{\mathcal{W}}$  is invertible, positive and  $AI \leq S_{\mathcal{W}} \leq BI$ . Any  $f \in \mathcal{H}$  has the representation  $f = \sum_{i \in I} v_i^2 S_{\mathcal{W}}^{-1} P_{W_i}(f)$ .

Now, we set the definition of perturbation of a fusion frame which considered in [5] and it was proved that the fusion frames are stable under these perturbations.

**Definition 2.6.** [5] Let  $\{W_i\}_{i \in I}$  and  $\{\tilde{W}_i\}_{i \in I}$  be families of closed subspaces in  $\mathcal{H}$ , let  $\{v_i\}_{i \in I}$  be positive numbers,  $0 \leq \lambda_1, \lambda_2 \leq 1$ , and  $\varepsilon > 0$ . If

$$\|(P_{W_i} - P_{\tilde{W}_i})f\| \leq \lambda_1 \|P_{W_i}f\| + \lambda_2 \|P_{\tilde{W}_i}f\| + \varepsilon \|f\| \quad f \in \mathcal{H},$$

then, we say that  $\{(\tilde{W}_i, v_i)\}_{i \in I}$  is a  $(\lambda_1, \lambda_2, \varepsilon)$ -perturbation of  $\{(W_i, v_i)\}_{i \in I}$ .

A different notion of perturbation of fusion frames was introduced in [19], which is equivalent to the above notion of perturbation in finite dimensional case.

**Definition 2.7.** [19] Let  $\mu > 0$  and let  $\mathcal{W} = \{(W_i, w_i)\}_{i=1}^N$  and  $\mathcal{V} = \{(V_i, v_i)\}_{i=1}^N$  be two Bessel fusion sequences in  $\mathcal{H}$ . We say that  $\mathcal{V}$  is a  $\mu$ -perturbation of  $\mathcal{W}$  (and vice versa) if  $(v_i P_{V_i})_{i=1}^N$  is a  $\mu$ -perturbation of  $(w_i P_{W_i})_{i=1}^N$  or  $\|T_{\mathcal{W}} - T_{\mathcal{V}}\| \leq \mu$  and so

$$\|w_i P_{W_i} - v_i P_{V_i}\| \leq \mu.$$

We state the Proposition 5.2 from [5] for finite dimensional Hilbert space  $\mathcal{H}^n$  by assuming that  $\lambda_1 = \lambda_2 = 0$  and  $\varepsilon = \mu$ .

**Theorem 2.8.** [5] Let  $\mathcal{W} = \{W_i\}_{i=1}^N$  be a fusion frame for  $\mathcal{H}^n$  with bounds  $A, B$  and  $\mu > 0$  such that  $\sqrt{A} - \mu\sqrt{N} > 0$ . Further, let  $\{(V_i, v_i)\}_{i=1}^N$  be a  $\mu$ -perturbation of  $\mathcal{W}$ . Then  $\{(V_i, v_i)\}_{i=1}^N$  is a fusion frame with fusion frame bounds

$$(\sqrt{A} - \mu\sqrt{N})^2, \quad (\sqrt{B} + \mu\sqrt{N})^2.$$

One of our goals is to investigate the effect of perturbation of fusion frames on the redundancy.

**Definition 2.9.** [22] Let  $\mathcal{W} = \{(W_i, v_i)\}_{i=1}^N$  be a fusion frame for  $\mathcal{H}^n$  with bounds  $A$  and  $B$ . For each  $x \in \mathbb{S}$ , the *redundancy function*  $\mathcal{R}_{\mathcal{W}} : \mathbb{S} \rightarrow \mathbb{R}^+$  is defined by

$$\mathcal{R}_{\mathcal{W}}(x) = \sum_{i=1}^N \|P_{W_i}(x)\|^2, \quad x \in \mathbb{S}.$$

Notice that this notion is reminiscent of the definition of redundancy function for finite frames, in Definition 1.2, if  $\dim W_i = 1$  for  $i = 1, \dots, N$ .

**Definition 2.10.** For the fusion frame  $\mathcal{W} = \{(W_i, v_i)\}_{i=1}^N$ , the *upper redundancy* is defined by

$$\mathcal{R}_{\mathcal{W}}^+ = \sup_{x \in \mathbb{S}} \mathcal{R}_{\mathcal{W}}(x),$$

and the *lower redundancy* of  $\mathcal{W}$  by

$$\mathcal{R}_{\mathcal{W}}^- = \inf_{x \in \mathbb{S}} \mathcal{R}_{\mathcal{W}}(x).$$

We say that  $\mathcal{W}$  has *uniform redundancy* if  $\mathcal{R}_{\mathcal{W}}^- = \mathcal{R}_{\mathcal{W}}^+$ .

### 3. THE EFFECT OF PERTURBATIONS OF FRAMES ON THEIR REDUNDANCIES

In the definition of the redundancy function for the finite frame  $\Phi = \{\varphi_i\}_{i=1}^N$  the optimal bounds of the normalized version of  $\Phi$  considered as the lower and upper redundancies [2]. It is easy to prove the following lemma:

**Lemma 3.1.** *Let  $\Phi = \{\varphi_i\}_{i=1}^N$  be a frame for  $\mathcal{H}^n$  and  $\Psi = \{\psi_i\}_{i=1}^N$  be a  $\mu$ -perturbation of  $\Phi$  such that  $\|\varphi_i\| = \|\psi_i\|$  for all  $i = 1, \dots, N$ . Then the normalized version of  $\Psi$  is also a  $\mu$ -perturbation of the normalized version of  $\Phi$ .*

*Proof.* The sequence  $\Psi = \{\psi_i\}_{i=1}^N$  is a  $\mu$ -perturbation of the frame  $\Phi$ , so we have

$$\left\| \sum c_k (\varphi_i - \psi_i) \right\| \leq \mu \|c\|_{l^2}$$

for all finite sequence  $c = \{c_k\}$ . By assumption, we have  $\|\varphi_i\| = \|\psi_i\| = \alpha_i$  for  $i = 1, \dots, N$ . Hence for all finite scalar sequence  $c = \{c_k\}$ ,

$$\left\| \sum c_k \left( \frac{\varphi_i}{\|\varphi_i\|} - \frac{\psi_i}{\|\psi_i\|} \right) \right\| = \left\| \sum c_k \left( \frac{\varphi_i}{\alpha_i} - \frac{\psi_i}{\alpha_i} \right) \right\| = \left\| \sum c'_k (\varphi_i - \psi_i) \right\|.$$

By the definition of the perturbation of a frame we have

$$\left\| \sum c'_k (\varphi_i - \psi_i) \right\| \leq \mu \|c'\|_{l^2}.$$

Therefore, the normalized version of  $\Psi$  is a  $\mu$ -perturbation of the normalized version of  $\Phi$ .  $\square$

By using the above lemma, we get the following theorem.

**Theorem 3.2.** *Let  $\Phi = \{\varphi_i\}_{i=1}^N$  be a frame for  $\mathcal{H}^n$  with bounds  $0 < A \leq B$  and  $\mu < \sqrt{A}$ . Let  $\Psi = \{\psi_i\}_{i=1}^N$  be a  $\mu$ -perturbation of  $\Phi$  such that  $\|\varphi_i\| = \|\psi_i\|$  for all  $i = 1, \dots, N$ . Let  $\mathcal{R}_{\Phi}^-$  and  $\mathcal{R}_{\Phi}^+$  denote the lower and upper redundancies for the frame  $\Phi$ , respectively. Then  $\Psi$  is also a frame for  $\mathcal{H}^n$  and the lower and upper redundancies for the frame  $\Psi$  are as follows:*

$$\mathcal{R}_{\Psi}^- = \mathcal{R}_{\Phi}^- \left( 1 - \frac{\mu}{\sqrt{\mathcal{R}_{\Phi}^-}} \right)^2, \quad \mathcal{R}_{\Psi}^+ = \mathcal{R}_{\Phi}^+ \left( 1 + \frac{\mu}{\sqrt{\mathcal{R}_{\Phi}^+}} \right)^2.$$

Moreover, if  $\Phi = \{\varphi_i\}_{i=1}^N$  is a Riesz basis, then

$$\mathcal{R}_{\Phi}^- = \mathcal{R}_{\Psi}^- = \mathcal{R}_{\Psi}^+ = \mathcal{R}_{\Phi}^+ = 1.$$

*Proof.* Let  $\Phi = \{\varphi_i\}_{i=1}^N$  be a frame for  $\mathcal{H}^n$  with bounds  $0 < A \leq B$  and  $\mu < \sqrt{A}$ . Let  $\Psi = \{\psi_i\}_{i=1}^N$  be a  $\mu$ -perturbation of  $\Phi$ . Then by Theorem 2.3,  $\Psi$  is also a frame for  $\mathcal{H}^n$  with bounds

$$A(1 - \frac{\mu}{\sqrt{A}})^2, B(1 + \frac{\mu}{\sqrt{B}})^2.$$

By assumption, we have  $\|\varphi_i\| = \|\psi_i\| = \alpha_i$  for  $i = 1, \dots, N$ . Then by Lemma 3.1, the normalized version of  $\Psi$  is also a  $\mu$ -perturbation of the normalized version of  $\Phi$ . We know that the redundancy function [2] for a frame consider the normalized version of the frame and lower and upper redundancies coincide the optimal lower and upper frame bounds of the normalized version of the original frame. So the frame bounds of the frame  $\{\frac{\varphi_i}{\|\varphi_i\|}\}_{i=1}^N$  equal to  $\mathcal{R}_\Phi^-$  and  $\mathcal{R}_\Phi^+$ . Similar statement holds for redundancies of the frame  $\Psi$ . Hence, we conclude

$$\mathcal{R}_\Psi^- = \mathcal{R}_\Phi^- (1 - \frac{\mu}{\sqrt{\mathcal{R}_\Phi^-}})^2, \quad \mathcal{R}_\Psi^+ = \mathcal{R}_\Phi^+ (1 + \frac{\mu}{\sqrt{\mathcal{R}_\Phi^+}})^2.$$

For the moreover part, we know [2] that the redundancy of each Riesz basis equals to one. Hence by Theorem 2.3 the sequence  $\Psi = \{\psi_i\}_{i=1}^N$  is a Riesz basis, so

$$\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+ = \mathcal{R}_\Psi^- = \mathcal{R}_\Psi^+ = 1.$$

□

By extra hypothesis  $\mathcal{R}_\Phi^-, \mathcal{R}_\Phi^+ < \infty$  on the lower and upper redundancies of the frame  $\Phi$ , above theorem extended to the infinite case. We can express the lower and upper redundancies of a frame in terms of cosine angle between some subspaces. At the first we review the definition of sine and cosine between subspaces [15].

**Definition 3.3.** [15] Assuming that  $V$  and  $W$  are subspaces of  $\mathcal{H}$  and  $V \neq 0$ . The angle from  $V$  to  $W$  is defined as the unique number  $\theta(V, W) \in [0, \frac{\pi}{2}]$  for which

$$(3.1) \quad \cos \theta(V, W) = \inf_{f \in V, \|f\|=1} \|P_W(f)\|.$$

We will frequently use the notion

$$R(V, W) = \cos \theta(V, W),$$

which is called the infimum cosine angle. If we take the supremum instead of the infimum of the right hand side of (3.1), we obtain the supremum cosine angle  $S(V, W)$  of  $V$  and  $W$ , that is

$$S(V, W) = \sup_{f \in V, \|f\|=1} \|P_W(f)\|.$$

$R$  and  $S$  are related by

$$R(V, W) = (1 - S^2(V, W^\perp))^{\frac{1}{2}}.$$

The gap of subspace  $V$  to  $W$  is defined by

$$\delta(V, W) = \sup_{x \in V, \|x\|=1} \text{dist}(x, W) = \sup_{f \in V, \|f\|=1} \inf_{y \in W} \|x - y\|.$$

An elementary calculation shows that the gap of  $V$  to  $W$  are related via

$$\delta(V, W) = \sin \theta(V, W).$$

By the properties of *inf* and *sup* and applying the definitions of infimum cosine and supremum cosine in the Definition 3.3, we have this elegant relations:

$$\mathcal{R}_{\Phi}^{-} = \sum_{i=1}^N R^2(W', \langle \varphi_i \rangle), \quad \mathcal{R}_{\Phi}^{+} = \sum_{i=1}^N S^2(W', \langle \varphi_i \rangle),$$

where  $W'$  is a subspace of  $\mathcal{H}$  which contains the unit sphere  $\mathbb{S}$ , i.e.,  $\mathbb{S} \subseteq W'$ . Therefore in the Theorem 3.2, we have

$$\mathcal{R}_{\Psi}^{-} = \sum_{i=1}^N R^2(W', \langle \varphi_i \rangle) \left(1 - \frac{\mu}{\sqrt{\sum_{i=1}^N R^2(W', \langle \varphi_i \rangle)}}\right)^2,$$

and for upper redundancy,

$$\mathcal{R}_{\Psi}^{+} = \sum_{i=1}^N S^2(W', \langle \varphi_i \rangle) \left(1 + \frac{\mu}{\sqrt{\sum_{i=1}^N S^2(W', \langle \varphi_i \rangle)}}\right)^2.$$

The above relations can be expressed in terms of the gap between the subspaces  $W'$  and  $\langle \varphi_i \rangle$ , because we have

$$\delta(W', \langle \varphi_i \rangle) = \sqrt{1 - R^2(W', \langle \varphi_i \rangle)}.$$

#### 4. THE EFFECT OF PERTURBATIONS OF FUSION FRAMES ON THEIR REDUNDANCIES

In the definition of the redundancy function for fusion frames [22], we consider lower and upper bounds of the normalized version of a fusion frame as lower and upper redundancies.

In the following theorem we investigate the relationship between the redundancy of a fusion frame and the redundancy of its perturbations. We assume that each subspace has a weight equal 1.

**Theorem 4.1.** *Let  $\mathcal{W} = \{W_i\}_{i=1}^N$  be a fusion frame for  $\mathcal{H}^n$  with bounds  $A, B$  and  $\mu > 0$  such that  $\sqrt{A} - \mu\sqrt{N} > 0$ . Let  $\mathcal{V} = \{V_i\}_{i=1}^N$  be a  $\mu$ -perturbation of  $\mathcal{W}$ . Then  $\mathcal{V}$  is a fusion frame for  $\mathcal{H}^n$  and the following relations between lower and upper redundancies of  $\mathcal{W}$  and  $\mathcal{V}$  hold:*

$$\mathcal{R}_{\mathcal{V}}^{-} = (\sqrt{\mathcal{R}_{\mathcal{W}}^{-}} - \mu\sqrt{N})^2, \quad \mathcal{R}_{\mathcal{V}}^{+} = (\sqrt{\mathcal{R}_{\mathcal{W}}^{+}} + \mu\sqrt{N})^2.$$



*Proof.* The family  $\mathcal{V}$  is a  $\mu$ -perturbation of the fusion frame  $\mathcal{W}$ . Hence by Theorem 2.8,  $\mathcal{V}$  is also a fusion frame for  $\mathcal{H}^n$ . The subspaces in fusion frames  $\mathcal{W}$  and  $\mathcal{V}$  have weights equal to 1, so they are normalized fusion frames and therefore the optimal bounds are equal to lower and upper redundancies. Theorem 2.8 implies:

$$\mathcal{R}_{\mathcal{V}}^{-} = (\sqrt{\mathcal{R}_{\mathcal{W}}^{-}} - \mu\sqrt{N})^2, \quad \mathcal{R}_{\mathcal{V}}^{+} = (\sqrt{\mathcal{R}_{\mathcal{W}}^{+}} + \mu\sqrt{N})^2.$$

□

Let  $\mathcal{W} = \{W_i\}_{i=1}^N$  be a fusion frame for  $\mathcal{H}^n$ . The lower and upper redundancies of  $\mathcal{W}$  can be express as follows:

Let  $W'$  be a subspace of  $\mathcal{H}^n$  such that involves the unit sphere  $\mathbb{S}$ , i.e.,  $\mathbb{S} \subseteq W'$ . Then

$$\mathcal{R}_{\mathcal{W}}^{-} = \sum_{i=1}^N R^2(W', W_i), \quad \mathcal{R}_{\mathcal{W}}^{+} = \sum_{i=1}^N S^2(W', W_i).$$

**Acknowledgments:** The authors would like to thank referee(s) for valuable comments and suggestions.

## REFERENCES

- [1] R. Balan, *Stability theorems for Fourier frames and wavelet Riesz bases*, J. Fourier Anal. Appl. 3 (1997), 499-504.
- [2] B. G. Bodmann, P. G. Casazza and G. Kutyniok, *A quantitative notion of redundancy for finite frames*, Appl. Comput. Harmon. Anal. 30 (2011), 348-362.
- [3] A. M. Bruckstein, D. L. Donoho and M. Elad, *From sparse solutions of systems of equations to sparse modeling of signals and images*, SIAM Rev. 51 (2009), 34-81.
- [4] P. Casazza and G. Kutyniok, *Finite Frames Theory and Applications*, Applied and Numerical Harmonic Analysis. Boston, MA: Birkhauser, 2013.
- [5] P. G. Casazza, G. Kutyniok and S. Li, *Fusion frames and distributed processing*, Appl. Comput. Harmon. Anal. 25 (2008), 114-132.
- [6] P. G. Casazza and G. Kutyniok, *Frames of subspaces*, in: Wavelets, Frames and Operator Theory (College Park, MD, 2003), Contempt. Math. 345, Amer. Math. Soc. Providence, RI. (2004), 87-113.
- [7] P. G. Casazza and J. C. Tremain, *The Kadison-Singer problem in mathematics and engineering*, Proc. Natl. Acad. Sci. 103 (2006), 2032-2039.
- [8] O. Christensen, *An introduction to frames and Riesz bases*, Birkhäuser, Boston 2003.
- [9] O. Christensen and C. Heil, *Perturbation of Banach frames and atomic decompositions*, Math. Nachr. 185 (1997), 33-47.
- [10] R. Duffin and A. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. 72 (1952), 341-366.
- [11] Y. C. Eldar and G. D. Forney, *Optimal tight frames and quantum measurement*, IEEE Trans. Inform. Theory 48 (2002), 599-610.
- [12] A. Ghaani Farashahi, *Wave packet transforms over finite cyclic groups*, Linear Algebra Appl. 489 (2016), 75-92.
- [13] 3. A. Ghaani Farashahi, *Wave packet transform over finite fields*, Electronic Journal of Linear Algebra, Vol. 30, (2015), 507-529 .
- [14] A. Ghaani Farashahi, *Cyclic wave packet transform on finite Abelian groups of prime order*, Int. J. Wavelets Multiresolut. Inf. Process. 12 (6) (2014), 1450041 (14 pages).

- [15] S. Heineken, E. Mathusika and V. Paternostro, *Perturbed frame sequences: Canonical dual systems, approximate reconstructions and applications*, Int. J. Wavelets Multiresolut. Inf. Process 12(2) (2014), 1450019 (19 pages)
- [16] V. Kaftal, D. R. Larson and S. Zhang, *Operator-valued frames*, Trans. Amer. Math. Soc. 361 (2009), 6349-6385.
- [17] G. Kutyniok, J. Lemvig and W. Q. Lim, *Optimally sparse approximations of 3D functions by compactly supported shearlet frames*, SIAM J. Math. Anal. 44 (2012), 2962-3017.
- [18] G. Kutyniok, V. Mehrmann and P. Petersen, *Regularization and numerical solution of the inverse scattering problem using shearlet frames*, Journal of Inverse and Ill-posed Problems, May 2016.
- [19] G. Kutyniok, F. Philipp and V. Paternostro, *The effect of perturbations of frame sequences and fusion frames on their duals*, arXiv:1509.04160, 2015.
- [20] L. Oeding, E. Robeva and B. Sturmfels, *Decomposing tensors into frames*, Adv. Appl. Math. vol. 73 (2016), 125-153.
- [21] G. Pfander, *Gabor Frames in Finite Dimensions*, In G. E. Pfander, P. G. Casazza, and G. Kutyniok, editors, Finite Frames, Applied and Numerical Harmonic Analysis, 193-239. Birkhauser Boston, 2013.
- [22] A. Rahimi, G. Zandi and B. Daraby, *Redundancy of fusion frames in Hilbert spaces*, Complex Anal. Oper. Theory 10 (2016), 545-565.
- [23] W. Sun, *g-frames and g-Riesz bases*, J. Math. Anal. Appl. 322 (2006), 437-452.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARAGHEH, P. O. BOX 55181-83111, MARAGHEH, IRAN.

*E-mail address:* rahimi@maragheh.ac.ir

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARAGHEH, P. O. BOX 55181-83111, MARAGHEH, IRAN.

*E-mail address:* zgolaleh@yahoo.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARAGHEH, P. O. BOX 55181-83111, MARAGHEH, IRAN.

*E-mail address:* bdaraby@maragheh.ac.ir